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On a zero speed sensitive cellular automaton *

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Abstract

Using an unusual, yet natural invariant measure we show that there exists a sensitive cellular automaton whose perturbations propagate at asymptotically null speed for almost all configurations. More specifically, we prove that Lyapunov Exponents measuring pointwise or average linear speeds of the faster perturbations are equal to zero. We show that this implies the nullity of the measurable entropy. The measure μ we consider gives the μ-expansiveness property to the automaton. It is constructed with respect to a factor dynamical system based on simple "counter dynamics". As a counterpart, we prove that in the case of positively expansive automata, the perturbations move at positive linear speed over all the configurations.

1 Introduction

A one-dimensional cellular automaton is a discrete mathematical idealization of a space-time physical system. The space $A^{\mathbb{Z}}$ on which it acts is the set of doubly infinite sequences of elements of a finite set A; it is called the *configuration space*. The discrete time is represented by the action of a cellular automaton F on this space. Cellular automata are a class of dynamical

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systems on which two different kinds of measurable entropy can be considered: the entropy with respect to the shift σ (which we call spatial) and the entropy with respect to F (which we call temporal). The temporal entropy depends on the way the automaton "moves" the spatial entropy using a local rule on each site of the configuration space. The propagation speed of the different one-sided configurations, also called perturbations in this case, can be defined on a specific infinite configuration, or as an average value on the configuration space endowed with a probability measure. We can consider perturbations moving from the left to the right or from the right to the left side of the two sided sequences. Here we prove that the perturbations (going to the left or to the right) move at a positive speed on all the configurations for a positively expansive cellular automata and that in the sensitive case, there exist automata with the property that for almost all the configurations, the perturbations can move to infinity but at asymptotically null speed.

Cellular automata can be roughly divided into two classes: the class of automata which have *equicontinuous points* and the class of *sensitive* cellular automata (Kůrka introduces a more precise classification in [6]). This partition of CA into ordered ones and disordered ones also corresponds to the cases where the perturbations cannot move to infinity (equicontinuous class) and to the cases where there always exist perturbations that propagate to infinity.

The existence of equicontinuity points is equivalent to the existence of a so-called "blocking word" (see [6], [8]). Such a word stops the propagation of perturbations so roughly speaking in the non sensitive case, the speed of propagation is equal to zero for all the points which contains infinitely many occurrences of "blocking words". For a sensitive automaton there is no such word, so that perturbations may go to infinity. However, there are few results about the speed of propagation of perturbations in sensitive cellular automata except for the positively expansive subclass.

In [7], Shereshevsky gave a first formal definition of these speeds of propagation (for each point and for almost all points) and called them Lyapunov exponents because of the analogy (using an appropriate metric) with the well known exponents of the differentiable dynamical systems.

Here we use a second definition of these discrete Lyapunov exponents given by Tisseur [8]. The two definitions are quite similar, but for each point, the Shereshevsky's one uses a maximum value on the shift orbit. For this reason the Shereshevsky's exponents (pointwise or global) can not see the "blocking words" of the equicontinuous points and could give a positive value (for almost all points) to the speed on these points. Furthemore, in our sensitive example (see Section 5), for almost all infinite configurations, there always exists some increasing (in size) sequences of finite configurations where the speed of propagation is not asymptoticaly null which implies that the initial definition gives a positive value to the speed and does not take into account a part of the dynamic of the measurable dynamical systems.

Using the initial definition of Lyapunov exponents due to Shereshevsky [7], Finelli, Manzini, Margara ([2]) have shown that positive expansiveness implies positivity of the Shereshevski pointwise Lyapunov exponents at all points.

Here we show that the statement of Finelli, Manzini, Margara still holds for our definition of pointwise exponents and the main difference between the two results is that we obtain that the exponents are positive for all points using a liminf rather that a limsup.

Proposition 1 *For a positively expansive cellular automaton F acting on $A^{\mathbb{Z}}$, there is a constant $\Lambda > 0$ such that, for all $x \in X$,*

$$\lambda^+(x) \geq \Lambda \text{ and } \lambda^-(x) \geq \Lambda,$$

where $\lambda^+(x)$ and $\lambda^-(x)$ are respectively the right and left pointwise Lyapunov exponents.

The first part of the proof uses standard compactness arguments. The result is stronger and the proof is completely different from the one in [2]. This result is called Proposition 2 in Section 3 and it is stated for all F-invariant subshifts X.

Our main result concerns sensitive automata and average Lyapunov exponents (I_μ^+ and I_μ^-). We construct a sensitive cellular automaton F and a (σ, F)-invariant measure μ^F such that the average Lyapunov exponents $I_{\mu^F}^{\pm}$ are equal to zero.

By showing (see Proposition 3) that the nullity of the average Lyapunov exponents implies that the measurable entropy is equal to zero, we obtain that our particular automaton have null measurable entropy $h_{\mu^F}(F) = 0$.

We also prove that this automaton is not only sensitive but μ^F-expansive which is a measurable equivalent to positive expansiveness introduced by Gilman in [3]. So even if this automaton is very close to positive expansiveness (in the measurable sense), its pointwise Lyapunov exponents are equal to zero almost everywhere (using Fatou's lemma) for a "natural" measure μ^F with positive entropy under the shift. The μ^F-expansiveness means that "almost all perturbations" move to infinity and the Lyapunov exponents represent the speed of the faster perturbation, so in our example almost all perturbations move to infinity at asymptoticaly null speed.

In view of this example, Lyapunov exponents or average speed of perturbations appear a useful tool for proving that a cellular automata has zero measure-theoretic entropy.

The next statement gathers the conclusions of Proposition 3, Proposition 5, Lemma 3 , Proposition 6, Proposition 7, Corollary 1, Remark 6.

Theorem 1 *There exists a sensitive cellular automaton F with the following properties: there exists a (σ, F)-invariant measure μ^F such that $h_{\mu^F}(\sigma) > 0$ and the Lyapunov exponents are equal to zero, i.e., $I^{\pm}_{\mu^F} = 0$, which implies that $h_{\mu^F}(F) = 0$. Furthermore this automaton F has the μ^F-expansiveness property.*

Let us describe the dynamics of the cellular automaton F and the related "natural" invariant measure μ^F that we consider.

In order to have a perturbation moving to infinity but at a sublinear speed, we define a cellular automaton with an underlying "counters dynamics", i.e. with a factor dynamical system based on "counters dynamics".

Consider this factor dynamical system of F and call a "counter" of size L a set $\{0, \ldots, L-1\} \subset \mathbb{N}$. A trivial dynamic on this finite set is addition of 1 modulo L. Consider a bi-infinite sequence of counters (indexed by $\mathbb{Z}$). The sizes of the counters will be chosen randomly and unboundly. If at each time step every counter is increased by one, all counters count at their own rythm, independently. Now we introduce a (left to right) interaction between them. Assume that each time a counter reaches the top (or passes through 0; or is at 0), it gives an overflow to its right neighbour. That is, at each time, a counter increases by 1 unless its left neighbour reaches the top, in which case it increases by 2. This object is not a cellular automaton because its state space is unbounded. However, this rough definition should be enough to suggest the idea.

• The dynamics is sensitive. Choose a configuration, if we change the counters to the left of some coordinate $-s$, we change the frequency of apparition of overflows in the counter at position $-s$ (for example this happens if we put larger and larger counters). Then the perturbation eventually appears at the coordinate 0.

• The speed at which this perturbation propagates is controlled by the sizes of the counters. The time it takes to get through a counter is more or less proportional to the size of this counter (more precisely, of the remaining time before it reaches 0 without an overflow). So a good choice of the law of the sizes of the counters allows us to control this mean time. More specifically, we prove that that with high probability, information will move slowly. If the size of the counters were bounded the speed would remain linear.

In the cellular automaton F, we "put up the counters" horizontally: we replace a counter of size $L = 2^l$ by a sequence of l digits and we separate sequences of digits by a special symbol, say E. Between two Es the dynamics of a counter is replaced by an odometer with overflow transmission to the right. More precisely, at each step the leftmost digit is increased by 1 and overflow is transmitted.

Note that to model the action of a cellular automaton we need to introduce in the factor dynamics a countdown which starts when the counter reaches the top. The end of the countdown corresponds to the transmission of the overflow. When the countdown is running, the time remaining before the emission of the overflow does not depend on a possible overflow emmited by a neighbouring counter. Nevertheless the effect of this overflow will affect the start of the next countdown.

Finally, we construct an invariant measure based on Cesaro means of the sequence $(\mu \circ F^n)$ where μ is a measure defined thanks to the counter dynamic of the factor dynamical system .

2 Definitions and notations

2.1 Symbolic systems and cellular automata

Let A be a finite set or alphabet. Denote by A^* the set of all concatenations of letters in A. $A^{\mathbb{Z}}$ is the set of bi-infinite sequences $x = (x_i)_{i\in\mathbb{Z}}$ also called *configuration space*. For $i \leq j$ in $\mathbb{Z}$, we denote by $x(i, j)$ the word $x_i \dots x_j$ and by $x(p, \infty)$ the infinite sequence $(v_i)_{i\in\mathbb{N}}$ with $v_i = x_{p+i}$. For $t \in \mathbb{N}$ and a word u we call *cylinder* the set $[u]_t = \{x \in A^{\mathbb{Z}} : x(t, t + |u|) = u\}$. The configuration set $A^{\mathbb{Z}}$ endowed with the product topology is a compact metric space. A metric compatible with this topology can be defined by the distance $d(x, y) = 2^{-i}$, where $i = \min\{|j| : \ x(j) \neq y(j)\}$.

The shift $\sigma\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x_i)_{i\in\mathbb{Z}} = (x_{i+1})_{i\in\mathbb{Z}}$. The dynamical system $(A^{\mathbb{Z}}, \sigma)$ is called the *full shift*. A *subshift* X is a closed shift-invariant subset X of $A^{\mathbb{Z}}$ endowed with the shift σ. It is possible to identify (X, σ) with the set X.

Consider a probability measure μ on the Borel sigma-algebra $\mathcal{B}$ of $A^{\mathbb{Z}}$. When μ is σ-invariant the *topological support* of μ is a subshift denoted by $S(\mu)$. We shall say that the topological support is trivial if it is countable. If $\alpha = (A_1, \dots, A_n)$ and $\beta = (B_1, \dots, B_m)$ are two partitions of X, we denote by $\alpha \vee \beta$ the partition $\{A_i \cap B_j, i = 1, \dots, n, \ j = 1, \dots, m\}$. Let $T : X \to X$ be a measurable continuous map on a compact set X. The metric *entropy* $h_\mu(T)$ of T is an isomorphism invariant between two μ-preserving transformations. Let

$H_\mu(\alpha) = \sum_{A\in\alpha} \mu(A)\log\mu(A)$, where α is a finite partition of X. The entropy of the finite partition α is defined as $h_\mu(T,\alpha) = \lim_{n\to\infty} 1/nH_\mu(\vee_{i=0}^{n-1}T^{-i}\alpha)$ and the entropy of (X,T,μ) as $h_\mu(T) = \sup_\alpha h_\mu(T,\alpha)$.

A *cellular automaton* is a continuous self-map F on $A^{\mathbb{Z}}$ commuting with the shift. The Curtis-Hedlund-Lyndon theorem [4] states that for every cellular automaton F there exist an integer r and a *block map* $f : A^{2r+1} \mapsto A$ such that $F(x)_i = f(x_{i-r},\ldots,x_i,\ldots,x_{i+r})$. The integer r is called the *radius* of the cellular automaton. If X is a subshift of $A^{\mathbb{Z}}$ and $F(X) \subset X$, then the restriction of F to X determines a dynamical system (X,F) called a cellular automaton on X.

2.2 Equicontinuity, sensitivity and expansiveness

Let F be a cellular automaton on $A^{\mathbb{Z}}$.

Definition 1 (Equicontinuity) A point $x \in A^{\mathbb{Z}}$ is called an equicontinuous point (or Lyapunov stable) if for all $\epsilon > 0$, there exists $\eta > 0$ such that

$$d(x,y) \le \eta \implies \forall i > 0,\, d(T^i(x), T^i(y)) \le \epsilon.$$

Definition 2 (Sensitivity) The automaton $(A^{\mathbb{Z}}, F)$ is sensitive to initial conditions (or sensitive) if there exists a real number $\epsilon > 0$ such that

$$\forall x \in A^{\mathbb{Z}},\, \forall\delta > 0,\, \exists y \in A^{\mathbb{Z}},\;\; d(x,y) \le \delta,\;\; \exists n \in \mathbb{N},\;\; d(F^n(x), F^n(y)) \ge \epsilon.$$

The next definition appears in [3] for a Bernouilli measure.

Definition 3 (μ-Expansiveness) The automaton $(A^{\mathbb{Z}}, F)$ is μ-expansive if there exists a real number $\epsilon > 0$ such that for all x in $A^{\mathbb{Z}}$ one has

$$\mu\left(\{y \in X : \forall i \in \mathbb{N},\, d(F^i(x), F^i(y)) \le \epsilon\}\right) = 0.$$

Notice that in [3] Gilman gives a classification of cellular automata based on the μ-expansiveness and the μ-equicontinuity classes.

Definition 4 (Positive Expansiveness) The automaton $(A^{\mathbb{Z}}, F)$ is positively expansive if there exists a real number $\epsilon > 0$ such that,

$$\forall (x,y) \in (A^{\mathbb{Z}})^2,\, x \ne y\,, \exists n \in \mathbb{N} \text{ such that } d(F^n(x), F^n(y)) \ge \epsilon.$$

Kůrka [6] shows that, for cellular automata, sensitivity is equivalent to the absence of equicontinuous points.

2.3 Lyapunov exponents

For all $x \in A^{\mathbb{Z}}$, the sets

$W_s^+(x) = \{y \in A^{\mathbb{Z}} : \forall i \geq s,\, y_i = x_i\}, \quad W_s^-(x) = \{y \in A^{\mathbb{Z}} : \forall i \leq s,\, y_i = x_i\},$

are called *right and left set of all the perturbations of* x, respectively.

For all integer n, consider the smallest "distance" in terms of configurations at which a perturbation will not be able to influence the n first iterations of the automaton:

$$\begin{aligned} I_n^-(x) &= \min\{s \in \mathbb{N} : \forall 1 \leq i \leq n,\, F^i(W_s^-(x)) \subset W_0^-(F^i(x))\}, \quad (1) \\ I_n^+(x) &= \min\{s \in \mathbb{N} : \forall 1 \leq i \leq n,\, F^i(W_{-s}^+(x)) \subset W_0^+(F^i(x))\}. \end{aligned}$$

We can now define the pointwise Lyapunov exponents by

$$\lambda^+(x) = \liminf_{n\to\infty} \frac{I_n^+(x)}{n}, \quad \lambda^-(x) = \liminf_{n\to\infty} \frac{I_n^-(x)}{n}.$$

For a given configuration x, $\lambda^+(x)$ and $\lambda^-(x)$ represent the speed to which the left and right faster perturbations propagate.

Definition 5 (Lyapunov Exponents) For a μ shift-invariant measure on $A^{\mathbb{Z}}$, we call average Lyapunov exponents of the automaton $(A^{\mathbb{Z}}, F, \mu)$, the constants

$$I_\mu^+ = \liminf_{n\to\infty} \frac{I_{n,\mu}^+}{n}, \quad I_\mu^- = \liminf_{n\to\infty} \frac{I_{n,\mu}^-}{n}, \qquad (2)$$

where

$$I_{n,\mu}^+ = \int_X I_n^+(x) d\mu(x), \quad I_{n,\mu}^- = \int_X I_n^-(x) d\mu(x).$$

Remark 1 *The sensitivity of the automaton* $(A^{\mathbb{Z}}, F, \mu)$ *implies that for all* $x \in A^{\mathbb{Z}}$, $(I_n^+(x) + I_n^-(x))_{n\in\mathbb{N}}$ *goes to infinity.*

3 Lyapunov exponents of positively expansive cellular automata

Similar versions of the next lemma appear in [1] and [6]. The proof of similar results in [2] using limsup is based on completely different arguments.

Lemma 1 *Let* F *be a positively expansive CA with radius* r *acting on a* F*-invariant subshift* $X \subset A^{\mathbb{Z}}$. *There exists a positive integer* N^+ *such that for all* x *and* y *in* X *that verify* $x(-\infty, -r-1) = y(-\infty, -r-1)$ *and* $F^m(x)(-r, r) = F^m(y)(-r, r)$ *for all integers* $0 < m \leq N^+$, *we have* $x(r, 2r) = y(r, 2r)$.

Proof Let B_n be the subset of $(x,y) \in X \times X$ such that $x(-\infty,-r-1) = y(-\infty,-r-1)$, $x(r,2r) \neq y(r,2r)$ and $F^m(x)(-r,r) = F^m(y)(-r,r)$ for all $m < n$. Each B_n is closed and $B_{n+1} \subset B_n$. Positive expansiveness of F implies $\lim_{n\to\infty} B_n = \emptyset$ (see [1]). Since X is a compact set, there is a positive integer N^+ such that $B_{N^+} = \emptyset$. □

Proposition 2 *For a positively expansive CA acting on a bilateral subshift X, there is a constant $\Lambda > 0$ such that for all $x \in X$, $\lambda^{\pm}(x) \geq \Lambda$.*

Proof We give the proof for $\lambda^-(x)$ only, the proof for $\lambda^+(x)$ being similar.

Let r be the radius of the automaton. According to Lemma 1, for any point $x \in X$ we obtain that if $y \in W^-_{-1}(x)$ is such that $F^i(y)(-r,r) = F^i(x)(-r,r)$ ($\forall\ 1 \leq i \leq N^+$) then y must be in $W^-_r(x) \subset W^-_0(x)$. From the definition of $I^-_{N^+}(x)$ in (1), this implies that $I^-_{N^+}(x) \geq 2r$. Lemma 1 applied N^+ times implies that for each $0 \leq i \leq N^+$, $F^n\left(F^i(x)\right)(-r,r) = F^n\left(F^i(y)\right)(-r,r)$ for all $0 \leq n \leq N^+$. It follows that $F^i(x)(r,2r) = F^i(y)(r,2r)$ ($\forall\ 0 \leq i \leq N^+$). Using Lemma 1 once more and shifting r coordinates of x and y yields $\sigma^r(x)(r,2r) = \sigma^r(y)(r,2r) \Rightarrow x(2r,3r) = y(2r,3r)$ so that $I^-_{2N^+}(x) \geq 3r$. Hence, for each integer $t \geq 1$, using Lemma 1, $N^+(t-1)!+1$ times yields $x\left(tr,(t+1)r\right) = y\left(tr,(t+1)r\right)$ and therefore $I_{tN^+}(x) \geq (t+1)r$. Hence for all $n \geq N^+$ and all $x \in X$, $I^-_n(x) \geq (\frac{n}{N^+}+1)r$, so that

$$\lambda^-(x) = \lim_{n\to\infty} \inf \frac{I^-_n(x)}{n} \geq \frac{r}{N^+}.$$

□

4 Lyapunov Exponents and Entropy

Let F be a cellular automaton acting on a shift space $A^{\mathbb{Z}}$ and let μ be a σ-ergodic and F-invariant probability measure. According to the inequality

$$h_\mu(F) \leq h_\mu(\sigma)(I^+_\mu + I^-_\mu)$$

proved in [8, Theorem 5.1], one has $I^+_\mu + I^-_\mu = 0 \Rightarrow h_\mu(F) = 0$. Here we extend this result to the case of a σ and F-invariant measure on a sensitive cellular automaton.

Proposition 3 *If F is a sensitive cellular automaton and μ a shift and F-invariant measure, $I^+_\mu + I^-_\mu = 0 \Rightarrow h_\mu(F) = 0$.*

Proof Let α be a finite partition of $A^{\mathbb{Z}}$ and $\alpha_n^m(x)$ be the element of the partition $\alpha \vee \sigma^{-1}\alpha \vee \ldots \sigma^{-n+1}\alpha \vee \sigma^1\alpha \ldots \vee \sigma^m\alpha$ which contains x. Using [8, Eq. (8)] we see that, for all finite partitions α,

$$h_\mu(F,\alpha) \leq \int_{A^{\mathbb{Z}}} \liminf_{n\to\infty} \frac{-\log \mu(\alpha_{I_n^-(x)}^{I_n^+(x)}(x))}{I_n^+(x)+I_n^-(x)} \times \frac{I_n^+(x)+I_n^-(x)}{n} d\mu(x). \tag{3}$$

Suppose that $I_\mu^+ + I_\mu^- = \liminf_{n\to\infty} \int_X n^{-1}(I_n^+(x)+I_n^-(x))d\mu(x) = 0$. From Fatou's lemma we have $\int_X \liminf_{n\to\infty} n^{-1}(I_n^+(x)+I_n^-(x))d\mu(x) = 0$. Since $n^{-1}(I_n^+(x)+I_n^-(x))$ is always a positive or null rational, there exists a set $S \subset A^{\mathbb{Z}}$ of full measure such that $\forall x \in S$ we have $\liminf_{n\to\infty} n^{-1}(I_n^+(x)+I_n^-(x)) = 0$. Since F is sensitive, for all points $x \in A^{\mathbb{Z}}$, we have $\lim_{n\to\infty} I_n^+(x)+I_n^-(x) = +\infty$ (see [8]) and the Shannon-McMillan-Breiman theorem (in the extended case of $\mathbb{Z}$ actions see [5]) tells us that

$$\int_{A^{\mathbb{Z}}} \liminf_{n\to\infty} -\frac{\log \mu(\alpha_{I_n^-(x)}^{I_n^+(x)}(x))}{I_n^+(x)+I_n^-(x)} d\mu = h_\mu(\sigma,\alpha).$$

Since for all n and x, $-\log \mu(\alpha_{I_n^-(x)}^{I_n^+(x)}(x)) > 0$, we deduce that for all $\epsilon > 0$ there is an integer $M_\epsilon > 0$ and a set $S_\epsilon \subset S$ with $\mu(S_\epsilon) > 1-\epsilon$ such that for all $x \in S_\epsilon$,

$$0 \leq \liminf_{n\to\infty} \frac{-\log \mu(\alpha_{I_n^-(x)}^{I_n^+(x)}(x))}{I_n^+(x)+I_n^-(x)} \leq M_\epsilon.$$

For all $x \in S_\epsilon$ we obtain

$$\phi(x) := \liminf_{n\to\infty} \frac{-\log \mu(\alpha_{I_n^-(x)}^{I_n^+(x)}(x))}{I_n^+(x)+I_n^-(x)} \times \frac{I_n^+(x)+I_n^-(x)}{n} = 0,$$

which implies $\int_{S_\epsilon} \phi(x)d\mu(x) = 0$. Using the monotone convergence theorem we deduce $\int_{A^{\mathbb{Z}}} \phi(x)d\mu(x) = 0$. It then follows from (3) that $h_\mu(F) = \sup_\alpha h_\mu(F,\alpha) = 0$. □

5 The cellular automaton and its natural factor

We define a cellular automaton for which the dynamic on a set of full measure is similar to the "counters dynamic" described in the introduction. The unbounded size counters are "simulated" by the finite configurations in the interval between two special letters "E". We will refer to these special symbols as "emitters". Between two E's, the dynamic of a counter is replaced by an odometer with overflow transmission to the right. We add "2" and "3" to $\{0; 1\}$ in the set of digits in order to have the sensitive dynamic of counters with overflows transmission. The states 2 and 3 are interpreted as "0 + an overflow to be sent", and "1 + an overflow to be sent". Using this trick, we let overflow move only one site per time unit. Notice that 3 is necessary because it may happen that a counter is increased by 2 units in one time unit.

5.1 The cellular automaton

We define a cellular automaton F from $A^{\mathbb{Z}}$ to $A^{\mathbb{Z}}$ with $A = \{0; 1; 2; 3; E\}$. This automaton is the composition $F = F_d \circ F_p$ of two cellular automata F_d and F_p. The main automaton F_d is defined by the local rule f_d

$$\begin{aligned} f_d(x_{i-2}x_{i-1}x_i) \quad = \quad & \mathbf{1}_E(x_i)x_i + \mathbf{1}_{\overline{E}}(x_i)\left(x_i - 2 \times \mathbf{1}_{\{2,3\}}(x_i) + \mathbf{1}_{\{2,3\}}(x_{i-1})\right) \\ & + \mathbf{1}_{\overline{E}}(x_i)\mathbf{1}_E(x_{i-1})\left(1 + \mathbf{1}_{\{2\}}(x_{i-2})\right), \end{aligned} \tag{4}$$

where $\overline{E} = A \setminus \{E\}$ and,

$$\mathbf{1}_S(x_i) = \begin{cases} 1 & \text{if } x_i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The automaton F_p is a "projection" on the subshift of finite type made of sequences having at least three digits between two "E" which is left invariant by F_d. Its role is simply to restrict the dynamics to this subshift. It can be defined by the local rule f_p

$$f_p(x_{i-3}, \ldots, x_i, \ldots x_{i+3}) = \mathbf{1}_{\overline{E}}(x_i)x_i + \mathbf{1}_E(x_i)x_i \times \prod_{\substack{j=-3 \\ j\neq 0}}^{3} \mathbf{1}_{\overline{E}}(x_{i+j}). \tag{5}$$

The dynamic of F is illustrated in Figure 1 for a particular configuration. The projection f_p prevents the dynamic of F to have equicontinuous points (points with "blocking words"EE") and simplifies the relationship between the cellular automaton and the model. The non-surjective cellular automaton acts surjectively on its ω-limit space X which is a non finite type subshift with a minimal distance of 3 digits between two "E". By definition, we have $X = \lim_{n\to\infty} \cap_{i=1}^{n} F^i(A^{\mathbb{Z}})$. The set X is rather complicated. We do not want to give a complete description. Some basic remarks may be useful for a better understanding of the results. Note that the Es do not change after the first iteration and that $x_i = 3$ implies $x_{i-1} = E$. We can show that the word

$$
\begin{array}{rcccccccccccc}
x = \dots & 0 & E & 1 & 1 & 0 & E & 0 & 2 & 2 & 2 & E & \dots \\
F(x) = \dots & 0 & E & 2 & 1 & 0 & E & 1 & 0 & 1 & 1 & E & \dots \\
F^2(x) = \dots & 0 & E & 1 & 2 & 0 & E & 2 & 0 & 1 & 1 & E & \dots \\
F^3(x) = \dots & 0 & E & 2 & 0 & 1 & E & 1 & 1 & 1 & 1 & E & \dots \\
F^4(x) = \dots & 0 & E & 1 & 1 & 1 & E & 2 & 1 & 1 & 1 & E & \dots \\
F^5(x) = \dots & 0 & E & 2 & 1 & 1 & E & 1 & 2 & 1 & 1 & E & \dots \\
F^6(x) = \dots & 0 & E & 1 & 2 & 1 & E & 2 & 0 & 2 & 1 & E & \dots \\
F^7(x) = \dots & 0 & E & 2 & 0 & 2 & E & 1 & 1 & 0 & 2 & E & \dots \\
F^8(x) = \dots & 0 & E & 1 & 1 & 0 & E & 3 & 1 & 0 & 0 & E & \dots \\
F^9(x) = \dots & 0 & E & 2 & 1 & 0 & E & 2 & 2 & 0 & 0 & E & \dots
\end{array}
\tag{6}
$$

Figure 1: An illustration of the dynamic of F defined by 5–4 on the configuration x, assuming that x is preceded by enough 0.

222 does not appear after the second iteration and that $F^i(x)(k, k+1) = 22$ only if $F^{i-1}(x)(k-2\colon k+1) \in \{2E21, 0E31, 1E31, 2E31\}$. According to the definition (4), the evolution of finite configurations without emitter "E" leads to sequences which contains only the digits "1" and "0". This is the dynamic of the emitter "E" with the overflows crossing the "Es" that maintain and move the letters "2" and "3". There is at most two letters "2" between two consecutive letters "E". A typical word between two "E" is of the form $E3uE$, $E2uE$, $E3u_12u_2E$ or $E2u_12u_2E$, where u, u_1 and u_2 are finite sequences of letters "0" and "1" (the words u_1 and u_2 can be empty). Notice that all possible words u, u_1, u_2 do not appear in X. For example, the word $E200E$ does not belong to the language of X.

As we want to study the dynamic on finite (but unbounded) counters, we define the set $\Omega \subset X$ of configurations with infinitely many E in both directions. This non compact set is obviously invariant by the dynamics. We are going to define a semi conjugacy between (Ω, F) and the model in the next section.

5.2 The natural factor

In order to make more intuitive the study of the dynamic of F and to define (see Section 7) a natural measure, we introduce the projection of this CA which is a continuous dynamical system that commutes with an infinite state 1-dimensional shift.

The word between two consecutive Es can be seen as a "counter" that overflows onto its right neighbour when it is full. At each time step E "emits" 1 on its right except when the counter on its left overflows: in this case there is a carry of 1 so the E "emits" 2 on its right.

$$
\begin{array}{rcccccc}
x & = & \ldots\ 0 & \overbrace{E}^{\text{emitter } i} & 0\ \ 0\ \ 2 & \overbrace{E}^{\text{emitter } i+1} & 1\ \ 1\ \ 0\ \ 0 & \overbrace{E}^{\text{emitter } i+2} & \ldots \\
F(x) & = & \ldots\ 0 & E & \underbrace{1\ \ 0\ \ 0}_{\text{counter } i} & E & \underbrace{3\ \ 1\ \ 0\ \ 0}_{\text{counter } i+1} & E & \ldots
\end{array}
$$

In what follows we call *counter* a triple (l, c, r), where l is the number of digits of the counter, c is its state and r is the overflow position in the counter. In Figure 1, in the first counter the countdown starts in $F^5(x)$ when the "2" is followed by "11". This "2" can propagate at speed one to the next emitter "E".

Recall that $\Omega \subset X$ is a set of configurations with infinitely many Es in both directions and has at least three digits between two Es. Define the sequence $(s_j)_{j\in\mathbb{Z}}$ of the positions of the Es in $x \in \Omega$ as follows:

$$
\begin{array}{rcl}
s_0(x) & = & \sup\{i \le 0 \ :\ x_i = E\}, \\
s_{j+1}(x) & = & \inf\{i > s_j(x) \ :\ x_i = E\} \quad \text{for } j \ge 0, \\
s_j(x) & = & \sup\{i < s_{j+1}(x) \ :\ x_i = E\} \quad \text{for } j < 0.
\end{array}
$$

Denote by $u = (u_i)_{i\in\mathbb{Z}} = (l_i, c_i, r_i)_{i\in\mathbb{Z}}$ a bi-infinite sequence of counters. Let $B = \mathbb{N}^3$ and σ_B be the shift on $B^{\mathbb{Z}}$. We are going to define a function φ from $\Omega \to B^{\mathbb{Z}}$. We set for all $i \in \mathbb{Z}$, $l_i(x) = s_{i+1}(x) - s_i(x) - 1$ and define $d_i(x) = \sum_{j=s_i+1}^{s_{i+1}-1} x_j 2^{j-s_i-1}$. We denote by $\overline{c}_i = 2^{l_i}$ the period of the counter (l_i, c_i, r_i).

For each $x \in \Omega$ and $i \in \mathbb{Z}$ we set $c_i(x) = d_i(x)$ modulo $\overline{c}_i(x)$ and we write

$$
r_i(x) = \begin{cases} l_i + 1 - \max\{j \in \{s_i + 1, \ldots, s_{i+1} - 1\} \ :\ x_j > 1\} + s_i & \text{if } d_i(x) \ge \overline{c}_i(x) \\ 0 & \text{otherwise.} \end{cases}
$$

For each $x \in \Omega$ we can define $\varphi(x) = (l_i(x), c_i(x), r_i(x))_{i\in\mathbb{Z}}$. Remark that since $r_i(x) \le l_i(x)$ and $c_i(x) \le 2^{l_i(x)}$, the set $\varphi(\Omega)$ is a strict subset $(\mathbb{N}^3)^{\mathbb{Z}}$.

On $\varphi(\Omega)$, we define a dynamic on the counters through a local function. First we give a rule for incrementation of the counters. For $a = 1$ or 2, we set

$$
\left\{
\begin{array}{lcll}
(l_i, c_i, 0) + a & = & (l_i, c_i + a, 0) & \text{if } c_i < \overline{c}_i - a \quad (R1) \\
(l_i, c_i, 0) + a & = & (l_i, c_i + a - \overline{c_i}, l_i) & \text{if } c_i + a \ge \overline{c}_i \quad (R2) \\
(l_i, c_i, r_i) + a & = & (l_i, c_i + a, r_i - 1) & \text{if } r_i > 0 \quad (R3).
\end{array}
\right.
$$

$$
\begin{array}{rcccc}
u = \dots & (3,3,0) & (4,0,0) & \dots \\
H(u) = \dots & (3,4,0) & (4,1,0) & \dots \\
H^2(u) = \dots & (3,5,0) & (4,2,0) & \dots \\
H^3(u) = \dots & (3,6,0) & (4,3,0) & \dots \\
H^4(u) = \dots & (3,7,0) & (4,4,0) & \dots \\
H^5(u) = \dots & (3,0,3) & (4,5,0) & \dots \\
H^6(u) = \dots & (3,1,2) & (4,6,0) & \dots \\
H^7(u) = \dots & (3,2,1) & (4,7,0) & \dots \\
H^8(u) = \dots & (3,3,0) & (4,9,0) & \dots \\
H^9(u) = \dots & (3,4,0) & (4,10,0) & \dots
\end{array}
$$

Figure 2: The dynamic in Figure 1 for the natural factor.

We define the local map h on $\mathbb{N}^3 \times \mathbb{N}^3$ by

$$h(u_{i-1}u_i) = u_i + (1 + \mathbf{1}_{\{r_{i-1}=1\}}(u_{i-1})),$$

where the addition must be understood following the incrementation procedure above, with $a = 1 + \mathbf{1}_{\{r_{i-1}=1\}}(u_{i-1})$. Let H be the global function on $(\mathbb{N}^3)^{\mathbb{Z}}$. This is a "cellular automaton on a countable alphabet".

Note that, the $(l_i)_{i\in\mathbb{Z}}$ do not move under iterations. At each step, the counter c_i is increased by a modulo $\overline{c}_i$ ($a = 1$ in general, while $a = 2$ if counter $i-1$ "emits" an overflow). When c_i has made a complete turn r_i starts to count down l_i, $l_i - 1 \dots 1, 0$; after l_i steps r_i reaches 0, indicating that (in the automaton) the overflow has reached its position.

Remark 2 *Notice that if $2l < 2^l$, we cannot have $c_i = 2^l - a$ and $r_i > 0$ since r_i is back to 0 before c_{i-1} completes a new turn. This technical detail is the reason why we impose a minimal distance 3 (more than the distance one required for the sensitivity condition) between two successive E.*

5.3 Semi conjugacy

Proposition 4 *We have the following semi conjugacy,*

$$
\begin{array}{ccc}
\Omega & \overset{F}{\longrightarrow} & \Omega \\
\downarrow \varphi & & \downarrow \varphi \\
\varphi(\Omega) & \overset{H}{\longrightarrow} & \varphi(\Omega)
\end{array}
$$

with

$$\varphi \circ F = H \circ \varphi.$$

Proof Let $x \in \Omega$. Denote $\varphi(x) = (l_i, c_i, r_i)_{i\in\mathbb{Z}}$, $x' = F(x)$ and $\varphi(x') = (l'_i, c'_i, r'_i)_{i\in\mathbb{Z}}$. We have to prove that $(l_i, c_i, r_i) + 1 + \mathbf{1}_{\{r_{i-1}=1\}} = (l'_i, c'_i, r'_i)$ where the addition satisfies the rules R_1, R_2, R_3.

First, we recall that the Es do not move so that $l'_i = l_i$.

Consider the first digit after the ith emitter E :$x'_{s_i+1} = x_{s_i+1} - 2 \times \mathbf{1}_{\{2,3\}}(x_{s_i+1}) + 1 + \mathbf{1}_{\{2\}}(x_{s_i-1})$. Clearly we have $r_{i-1} = 1$ if and only if $x_{s_i-1} = 2$ so $x'_{s_i+1} = x_{s_i+1} + 1 + \mathbf{1}_{\{r_{i-1}=1\}} - 2 \times \mathbf{1}_{\{2,3\}}(x_{s_i+1})$. For all $s_i + 2 \leq j \leq s_{i+1} - 1$, $x'_j = x_j - 2 \times \mathbf{1}_{\{2\}}(x_j) + \mathbf{1}_{\{2,3\}}(x_{j-1})$. Since $d_i(x) = \sum_{j=s_i+1}^{s_{i+1}-1} x_j 2^{j-s_i-1}$, if $x_{s_{i+1}-1} \neq 2$ then $d_i(x') = d_i(x) + 1 + \mathbf{1}_{\{r_{i-1}=1\}}$ and if $x_{s_{i+1}-1} = 2$ then $d_i(x') = d_i(x) + 1 + \mathbf{1}_{\{r_{i-1}=1\}} - 2 \times 2^{-l_i} = d_i(x) + 1 + \mathbf{1}_{\{r_{i-1}=1\}} - \overline{c_i}$. As $c'_i = d_i(x') \bmod \overline{c_i}$ then for all $x \in \Omega$ one has $c'_i = c_i + 1 + \mathbf{1}_{\{r_{i-1}=1\}} \bmod \overline{c_i}$.

It remains to understand the evolution of the overflow r_i. First, notice that if $d_i(x') = \overline{c_i} = 2^{l_i}$ then $x'(s_i, s_{i+1}) = E21^{(l_i-1)}E$ and if $d_i(x') = \overline{c_i} + 1 = 2^{l_i}+1$ then $x'(s_i, s_{i+1}) = E31^{(l_i-1)}E$. After l_i iteration of F, the configurations $E21^{(l_i-1)}E$ and $E31^{(l_i-1)}E$ have the form $Ew2E$. The maximum value taken by $d_i(x)$ is when $x(s_i, s_{i+1}) = Eu2E$ where $d_i(z) < 2l_i$ if $z(s_i, s_{i+1}) = Eu0E$. As noted in Remark 2, the counters, which have at least a size of 3, have not the time to make a complete turn during the countdown ($2l_i < \overline{c_i}$) which implies that $d_i(x) \leq \overline{c_i} + 2l_i < 2\overline{c_i}$. Since each counters can not receive an overflow at each iteration then $d_i(x) < 2\overline{c_i} - 2$ (when $l_i \geq 3$, $l_{i-1} \geq 3$, $\theta \times l_i < \overline{c_i} - 2 = 2^{l_i} - 2$ where $\theta < 2$). Clearly $r_i = 0$ if and only if $d_i = c_i < \overline{c_i}$ (addition rule R_1). As $d_i < 2\overline{c_i} - 2$, if $c_i = \overline{c_i} - 2 = d_i(x)$ and $x_{s_i-1} = 2$ or $c_i = \overline{c_i} - 1 = d_i(x)$ ($r_i = 0$) then $x'(s_i, s_{i+1}) = E21^{(l_i-1)}E$ or $x'(s_i, s_{i+1}) = E31^{(l_i-1)}E$ which implies that $r'_i = l_i$ (addition rule R_2).

Now remark that if $r_i > 0$ then $x(s_i, s_{i+1}) = Eu21^kE$ with $0 \leq k \leq l_i - 1$ and u is a finite sequence of letters "0", "1", "2" or "3". Using the local rule of F, we obtain that the letter "2" move to the right of one coordinate which implies that $r'_i = r_i - 1$ (addition rule R_3).

□

Remark 3 *We remark that φ is not injective. Consider the subset of X defined by*

$$\Omega^* = \Omega \cap \{x \in \{0, 1, E\}^{\mathbb{Z}} \ : \ x_0 = E\}.$$

It is clear that φ is one to one between Ω^* and $\varphi(\Omega)$ since the origin is fixed and there is only one way to write the counters with 0 *and* 1. *We will use this set, keeping in mind that it is not invariant for the cellular automaton F.*

Remark 4 *If $x \in A^{\mathbb{Z}}$, we use $c_i(x)$ to denote $c_i(\varphi(x))$ and $l_i(x)$ instead of $l_i(\varphi(x))$. This should not yield any confusion. To take the dynamics into account, we write $c_i^t(x) = c_i(\varphi(F^t(x))) = c_i(H^t(\varphi(x)))$ and similarly for $l_i(x)$ and $r_i(x)$. Note that $l_i^t(x) = l_i(x)$ for all t.*

5.4 Limit periods

The natural period of the counter i, is its number of states, say $\overline{c}_i$. We introduce the notion of *real period* or *asymptotic period* of a counter which is, roughly speaking, the time mean of the successive observed periods. It can be formally defined as the inverse of the number of overflow emitted by the counter (to the right) per unit time. More precisely, we define $N_i(x)$, the inverse of the real period $p_i(x)$ for the counter number i in x by

$$N_i(x) = \lim_{t\to+\infty} \frac{1}{t} \sum_{k=0}^{t} \mathbf{1}_{\{1\}}(r_i^t(x)).$$

Lemma 2 *The real period $p_i(x)$ exists and, $N_i(x) = \frac{1}{p_i(x)} = \sum_{k=i}^{-\infty} 2^{-\sum_{j=i}^{k} l_j(x)}$.*

Proof Let n_i^t be the number of overflows emitted by the counter i in x before time t.

$$n_i^t(x) = \sum_{k=0}^{t} \mathbf{1}_{\{1\}}(r_i^t(x)).$$

The number of turns per unit time is the limit when $t \to +\infty$ of n_i^t/t if it exists. We can always consider the limsup N_i^+ and the liminf N_i^- of these sequences.

After t iterations, the counter indexed by i has been incremented by t (one at each time step) plus the number of overflow "received" from the counter $(i-1)$. We are looking for a recursive relationship between n_i^t and n_{i-1}^t. Some information is missing about the delays in the "overflow transmission", but we can give upper and lower bounds.

The number of overflows emitted by counter i at time t is essentially given by its initial position + its "effective increase" - the number of overflows delayed, divided by the size $\overline{c}_i$ of the counter. The delay in the overflow transmission is at least l_i. The initial state is at most $\overline{c}_i$. Hence, we have,

$$n_i^t(x) < \frac{\overline{c}_i(x) + t + n_{i-1}^t(x)}{\overline{c}_i(x)} \quad \text{and,} \quad n_i^t(x) > \frac{t + n_{i-1}^t(x) - l_i(x)}{\overline{c}_i(x)}.$$

So for the limsup N_i^+ and liminf N_i^-, we obtain $N_i^{\pm} = \frac{1}{\overline{c}_i}(1 + N_{i-1}^{\pm})$. Remark that for all $t \in \mathbb{N}$ we have $N_i^+ - N_i^- = \frac{1}{\overline{c}_i}(N_{i-1}^+ - N_{i-1}^-) = (\frac{1}{\overline{c}_i})(N_{i-t}^+ - N_{i-t}^-)$. Since $\frac{1}{\overline{c}_i} \leq N_i^- \leq N_i^+ \leq \frac{2}{\overline{c}_i}$, the limit called N_i exists and finally we have

$$N_i(x) = \sum_{k=i}^{-\infty} \prod_{j=i}^{k} \overline{c}_j^{-1}(x) = \sum_{k=i}^{-\infty} 2^{-\sum_{j=i}^{k} l_j(x)}. \tag{7}$$

□

Remark 5 *The series above are lesser than the convergent geometric series $(\sum_{k=1}^{n} 2^{-3k})_{n\in\mathbb{N}}$ since l_i is always greater than 3. Note that in the constant case, $l_i = L$, the limit confirm the intuition because the period is*

$$p = \frac{1}{N_i} = \frac{1}{\sum_{k=1}^{+\infty} 2^{-kL}} = 2^L - 1.$$

6 Sensitivity

For the special cellular automata F, we say that a measure μ satisfies conditions (*) if for all $l \in (\mathbb{N})^{\mathbb{Z}}$ one has $\mu(A^{\mathbb{Z}} \setminus \Omega) = 0$ and $\mu(\{x \in \Omega : (l_i(x))_{i\in\mathbb{Z}} = l\}) = 0$. These "natural" conditions are satisfied by the invariant measure μ^F we consider (see Section 7, Remark 6).

Proposition 5 *The automaton F is sensitive to initial conditions. Moreover, it is μ-expansive if μ satisfies condition (*).*

Proof Fix $\epsilon = 2^{-2} = \frac{1}{4}$ as the sensitive and μ-expansive constant. When $x \in \Omega$, it is possible to define $l(x)$ which is the sequence of the size of the counters for x and to use the model $(H, \varphi(\Omega))$ to understand the dynamic.

We can use Lemma 2 to prove that for the model, if we modify the negative coordinates of the sequence l_i we also modify the asymptotic behaviour of the counter at 0. Such a change in the asymptotic behaviour implies that at one moment, the configuration at 0 must be different. From the cellular automaton side, we will show that a change of the real period of the "central counter" will affect the sequences $(F^t(x)(-1,1))_{t\in\mathbb{N}}$ which is enough to prove the sensitivity and μ-expansiveness conditions.

For each $x \in \Omega$ with $x_0 \neq E$, consider the sequence $l_{[0,-\infty]}(x) = l_0(x)l_{-1}(x) \ldots l_{-k}(x) \ldots$. We claim that if $l_{[0,-\infty]}(x) \neq l_{[0,-\infty]}(y)$ then $N_0(x) \neq N_0(y)$. Let j be the first negative or null integer such that $l_j(x) \neq l_j(y)$. By Lemma 2, there exists a positive real $K_1 = \sum_{k=0}^{j-1} 2^{-\sum_{i=0}^{k} l_i(x)}$

such that

$$N_0(x) = K_1 + K_1 2^{-l_j(x)} + K_1 2^{-l_j(x)}(\sum_{k=j+1}^{\infty} 2^{-\sum_{i=j+1}^{k} l_i(x)})$$

and

$$N_0(y) = K_1 + K_1 2^{-l_j(y)} + K_1 2^{-l_j(y)}(\sum_{k=j+1}^{\infty} 2^{-\sum_{i=j+1}^{k} l_i(y)}).$$

So writing $K_2 = K_1 2^{-l_j(x)}$ we obtain

$$N_0(x) - N_0(y) > \frac{K_2}{2} - \frac{K_2}{2}(\sum_{k=j+1}^{\infty} 2^{-\sum_{i=j+1}^{k} l_i(y)}).$$

As $\sum_{k=i+1}^{\infty} 2^{-\sum_{j=i+1}^{k} l_j(y)}$ is less than the geometric series $\sum_{k=0}^{\infty} 2^{-3k} = \frac{1}{7}$ we prove the claim.

If $x_0 = E$, using the shift commutativity of F we obtain that if $l_{[-1,-\infty]}(x) \neq l_{[-1,-\infty]}(y)$ then $N_{-1}(x) \neq N_{-1}(y)$.

Remark that for each $x \in \Omega$ and $\delta = 2^{-n} > 0$, there exists $y \in \Omega$ such that $y_i = x_i$ for $i \geq -n$ and $l(x) \neq l(y)$ (sensitivity conditions). We are going to show that if $l_i(x) \neq l_i(y)$ ($i < 0$; $x \in \Omega$) then there exist some $t \in \mathbb{N}$ such that $F^t(x)(-1, 1) \neq F^t(y)(-1, 1)$.

First, we consider the case where $x_0 = E$. In this case, since $N_{-1}(x) \neq N_{-1}(y)$, there is t such that $n_{-1}^t(x) \neq n_{-1}^t(y)$. At least for the first such t, $r_{-1}^t(x) \neq r_{-1}^t(y)$ since $r_{-1}^t(x) = 1$ and $r_{-1}^t(y) \neq 1$. But this exactly means that $x_{-1}^t = 2$ and $y_{-1}^t \neq 2$. Hence $x_{-1}^t \neq y_{-1}^t$.

Next assume that $x_0 \neq E$. We have $N_0(x) > N_0(y)$. This implies that $n_0^t(x) - n_0^t(y)$ goes to infinity. Hence the difference $c_0^t(x) - c_0^t(y)$ (which can move by 0, 1 or -1 at each step) must take (modulo $\overline{c}_0$) all values between 0 and $\overline{c}_0 - 1$. In particular, at one time t, the difference must be equal to 2^{-T_0-1}. If $x_{-1} = E$, then $T_0 = -1$ which implies that $x_0^t \neq y_0^t$. Otherwise, in view of the conjugacy, it means that

$$\sum_{i=T_0+1}^{T_1-1} x_i^t 2^{i-T_0-1} - \sum_{i=T_0+1}^{T_1-1} y_i^t 2^{i-T_0-1} = 2^{-T_0-1} (\text{modulo } \overline{c}_0),$$

that is, either $x_0^t \neq y_0^t$ or $x_0^t = y_0^t$ and $x_{-1}^t \neq y_{-1}^t$. We have proved the sensitivity of F for $x \in \Omega$.

To show that $x \in \Omega$ satisfies the μ-expansiveness condition, we need to prove that the set of points which have the same asymptotic period for the central counter is a set a measure zero. For each point x the set of all points y such that $F^t(x)(-1,1) = F^t(y)(-1,1)$ or $d(F^t(x), F^t(y)) \leq \frac{1}{4}$ $(t \in \mathbb{N})$, is denoted by $D(x, \frac{1}{4})$. Following the arguments of the proof of the sensitivity condition above, we see that every change in the sequence of letters "Es" in the left coordinates will affect the central coordinates after a while, so we obtain that $D(x, \frac{1}{4}) \subset \{y : l_i(y) = l_i(x) : i < 0\}$. Since μ satisfies condition (*), we have $\mu(D(x, \frac{1}{4})) = 0$ which is the condition required for the μ-expansiveness.

Now suppose that $x \in \{0, 1, 2, 3, E\}^{\mathbb{Z}} \setminus \Omega$. First notice that if there is at least one letter E in the left coordinates, the sequence $F^t(x)(-1,1)$ does not depend on the position or even the existence of a letter E in the right coordinates. Recall that after one iteration, the word 22 appears only directly after a letter E. So when there is at least one letter E in the negative coordinates of x, the arguments and the proof given for $x \in \Omega$ still work.

If there is no letter E in the negative coordinates of x, for any $\delta = 2^{-n}$, we can consider any $y \in \Omega$ such that $y_i = x_i$ if $-n \leq i \leq n$. For $x \in \{0, 1, 2\}^{\mathbb{Z}}$, the dynamic is only given by the letter 2 which move on sequences of 1 (see Figure 1). The sequence $F^t(x)(-1,1)$ can not behave like a counter because it is an ultimately stationary sequence (after a letter "2" pass over a "1" it remains a "0" that can not be changed) and the real period of the central counter of y is obviously strictly greater than 1. Then in this case again, the sequences $F^t(x)(-1,1)$ and $F^t(y)(-1,1)$ will be different after a while which satisfies the sensitivity condition. Since the only points z such that $F^t(z)(-1,1) = F^t(x)(-1,1)$ belong to the set of null measure $\{0, 1, 2, 3\}^{\mathbb{Z}}$ (as μ satisfies conditions (*)), we obtain the μ-expansiveness condition. □

7 Invariant measures

7.1 Invariant measure for the model

We construct an invariant measure for the dynamics of the counters. Firstly, let ν_* denote a measure on $\mathbb{N} \setminus \{0, 1, 2\}$ and secondly, let $\nu = \nu_*^{\otimes \mathbb{Z}}$ be the product measure. To fix ideas, we take ν_* to be the geometric law of paramater $\nu = \frac{2}{3}$ on $\mathbb{N}$ conditionned to be larger than 3, i.e.,

$$\nu_*(k) = \begin{cases} \frac{\nu^k}{\sum_{j>2} \nu^j} & \text{if } k \geq 3 \\ 0 & \text{if } k < 3. \end{cases}$$

Notice that the expectation of l_0 is finite : $\sum_{l_0 > 2} \nu_*(l_0) \times l_0 < +\infty$. We denote by m_L the uniform measure on the finite set $\{0, \ldots, L-1\}$. Given a two-sided sequence $l = (l_i)_{i\in\mathbb{Z}}$, we define a measure on $\mathbb{N}^{\mathbb{Z}}$ supported on $\prod_{i\in\mathbb{Z}}\{0,\dots,2^{l_i}-1\}$, defining $m_l = \otimes_{i\in\mathbb{Z}} m_{2^{l_i}}$, so that for all $k_0,\dots,k_m$ integers,

$$m_{\mathbf{l}}(\{c_i = k_0,\dots,c_{i+m} = k_m\}) = \begin{cases} 2^{-\sum_{j=i}^{i+m} l_j} & \text{if, } \forall i \le j \le i+m,\ c_j < 2^{l_j} \\ 0 & \text{otherwise.} \end{cases}$$

We want this property for the counters to be preserved by the dynamics. But, for the overflow, we do not know a priori how the measure will behave. Next we construct an initial measure and iterate the sliding block code on the counters H. For all $l = (l_i)_{i\in\mathbb{Z}}$, we set $\nu_l = \otimes_{i\in\mathbb{Z}}\delta_{l_i}$, and, $\eta = \otimes_{i\in\mathbb{Z}}\delta_0$, where δ_k denotes the Dirac mass at integer k. We consider the measures

$$\mu_l^H = \nu_l \otimes m_l \otimes \eta,$$

which give mass 1 to the event $\{\forall i \in \mathbb{Z}, r_i = 0\}$. Then, we define,

$$\tilde{\mu}^H = \int_{\mathbb{N}^{\mathbb{Z}}} \mu_l^H \ \nu(dl).$$

The sequence $\frac{1}{n}\sum_{k=0}^{n-1}\tilde{\mu}^H \circ H^{-k}$ has convergent subsequences. We choose one of these subsequences, (n_i), and write

$$\mu^H = \lim_{i\to\infty}\frac{1}{n_i}\sum_{k=0}^{n_i-1}\tilde{\mu}^H \circ H^{-k}.$$

7.2 Invariant measure for F

Now we construct a shift and F-invariant measure for the cellular automaton space.

Recall that $\Omega^* = \Omega \cap \{x \in \{0,1,E\}^{\mathbb{Z}} \ : \ x_0 = E\}$ (see Remark 3). We write, for all integer k, $\Omega_k^* = \sigma^k\Omega^*$. Let us define, for all measurable subsets I of $A^{\mathbb{Z}}$,

$$\mu_l^F = \sum_{k=0}^{l_0-1}\mu_l^H\left(\varphi(I \cap \Omega_k^*)\right).$$

This measure distributes the mass on the l_0 points with the same image (in the model) corresponding to the l_0 possible shifts of origin. The total mass of this measure is l_0. Since the expectation of l is $\bar{l} = \sum_{i=3}^{\infty}\nu_*(l)\times l < \infty$ is finite (if $\nu = \frac{2}{3}$; $\bar{l} = 5$), we can define a probability measure

$$\tilde{\mu}^F = \frac{1}{\bar{l}}\int_{\mathbb{N}^{\mathbb{Z}}}\mu_l^F \ \nu(dl).$$

The measure $\tilde{\mu}^F$ is shift-invariant (see further) but it is supported on a non F-invariant set. In order to have a F-invariant measure we take an adherence value of the Cesaro mean. We choose a convergent subsequence (n_{i_j}) of the sequence (n_i) defining μ^H and write

$$\mu^F = \lim_{j\to\infty} \frac{1}{n_{i_j}} \sum_{k=0}^{n_{i_j}-1} \tilde{\mu}^F \circ F^{-k}.$$

Remark 6 *Since for the measure μ^F, the length between two "Es" follows a geometric law, the measure μ^F satisfies the conditions (*) defined in Section 6, and from Proposition 5, the automaton F is μ^F-expansive.*

Lemma 3 *The measure μ^H is σ_B and H-invariant. The measure μ^F is σ and F-invariant. For all measurable subsets U of $\varphi(\Omega)$ such that l_0 is constant on U,*

$$\mu^F(\varphi^{-1}(U)) = \frac{l_0}{\bar{l}}\mu^H(U).$$

Proof The shift invariance of μ^H follows from the fact that ν is a product measure and that the dynamic commutes with the shift so the Cesaro mean does not arm. H and F invariance of μ^H and μ^F follow from the standard argument on Cesaro means.

Shift invariance of μ^F comes from a classical argument of Kakutani towers because μ^H is essentially the induced measure of μ^F for the shift on the set Ω^*. We give some details. We choose a measurable set I such that l_0 is constant on I. Choose $l = (l_i)_{i\in\mathbb{Z}}$. Noticing that $\varphi(\sigma^{-1}(I) \cap \Omega_k^*) = \varphi(I \cap \Omega_k^*)$ as soon as $k \geq 1$, a decomposition of I such as $I = \cup_{k=0}^{l_0-1} I \cap \Omega_k^*$ yields

$$\mu_l^F(\sigma^{-1}I) \;\; = \;\; \mu_l^F(I) - \mu_l^F(I \cap \Omega^*) + \mu_l^F(\sigma^{-1}(I) \cap \Omega^*).$$

We remark that

$$\varphi(\sigma^{-1}(I) \cap \Omega^*) = \sigma_B^{-1}(\varphi(I \cap \Omega^*))$$

so that

$$\mu_l^F(\sigma^{-1}(I) \cap \Omega^*) = \mu_l^H\left(\sigma_B^{-1}(\varphi(I \cap \Omega^*))\right).$$

Now we integrate with respect to $\mathbf{l}$ and use the σ_B^{-1}-invariance of $\tilde{\mu}^H$ to conclude that

$$\tilde{\mu}^F(\sigma^{-1}I) = \tilde{\mu}^F(I).$$

For a measurable set I, we decompose $I = \cup_{L\in\mathbb{N}}(I \cap \{x \in \Omega : l_0(x) = L\})$. Since F commutes with σ, the Cesaro mean does not affect the shift invariance. Hence, μ^F is shift invariant.

Now, we make explicit the relationship between μ^H and μ^F. Let $V \subset \Omega$ be an event measurable with respect to the σ-algebra generated by $(l_i, c_i; i \in \mathbb{Z})$. There is an event $U \subset \varphi(\Omega)$ such that $V = \varphi^{-1}(U)$. Write $V = \cup_{L\in\mathbb{N}} V_L$, where $V_L = V \cap \{x \in \Omega : l_0(x) = L\}$. Since $\varphi(V_L \cap \Omega_k^*) = \varphi(V_L) =: U_L$ for all $0 \leq k < L$, we can write

$$\mu_l^F(V_L) = \sum_{k=0}^{L-1} \mu_l^H(\varphi(V_L \cap \Omega_k^*)) = L\mu_l^H(\varphi(V_L)).$$

Integrating with respect to l gives

$$\tilde{\mu}^F(V) = \frac{1}{\bar{l}} \sum_{L\in\mathbb{N}} L\tilde{\mu}^H(U_L)\nu(L).$$

If l_0 is constant on U then $L = l_0$ and we get

$$\tilde{\mu}^F(\varphi^{-1}(U)) = \frac{L}{\bar{l}}\tilde{\mu}^H(U)$$

Since $F^{-1}(\varphi^{-1}(U)) = \varphi^{-1}(H^{-1}(U))$ and the Cesaro means are taken along the same subsequences, we are done. $\square$

Lemma 4 *Fix $M \in \mathbb{Z}, M < 0$ and a sequence $L = (L_M, \ldots, L_0)$. Consider the cylinder $V_L = \{x \in \Omega : (l_M(x), \ldots, l_0(x)) = L\}$. For all $M < i < 0$, and all $0 \leq k < 2^{L_i}$,*

$$\mu^F(\{x : c_i(x) = k\}|V_L) = 2^{-L_i}.$$

Proof Let $\mathcal{F}$ be the σ-algebra generated by $\{(c_j, r_j)|j < 0\}$ and $\{l_j, j \in \mathbb{Z}\}$. We claim that, almost surely,

$$\mu^H(\{c_0 = k\} \mid \mathcal{F}) = m_l(\{c_0 = k\}) = 2^{-l_0}.$$

Let $u, v \in \varphi(\Omega)$. If $u_i = v_i$ for all $i < 0$, and $c_0(v) = c_0(u) + a$, then $c_0(H^n(v)) = c_0(H^n(u)) + a$. The function $\Delta_n(u) = c_0(H^n(u)) - c_0(u)$ is $\mathcal{F}$-measurable. We deduce that

$$\begin{aligned}
\tilde{\mu}^H(H^{-n}(\{c_0 = k\}) \ |\mathcal{F}) &= (\nu_{l(u)} \otimes m_{l(u)} \otimes \eta)(\{c_0 = k - \Delta_n(u)\} \mid \mathcal{F}) \\
&= m_{2^{l_0(u)}}(k - \Delta_n(u)) \\
&= m_{2^{l_0}}(k) \\
&= 2^{-l_0}.
\end{aligned}$$

Remark that the last claim implies $\mu^H(\{c_i = k\}|\varphi(V_L)) \geq \mu^H(H^{-n}(\{c_0 = k\}) \ |\mathcal{F})$, so $\mu^H(\{c_i = k\}|\varphi(V_L)) = 2^{-l_0}$.

We prove the claim by taking the Cesaro mean and by going to the limit. We extend the result to all integer i using the shift invariance and the independence of c_i with respect to l_j when $j > i$. We conclude by applying Lemma 3:

$$\begin{aligned}
\mu^F(\{x : c_i(x) = k\}|V_L) &= \frac{\mu^F(\{x : c_i(x) = k\} \cap V_L)}{\mu^F(V_L)} \\
&= \frac{\frac{L_0}{l}\mu^H(\varphi(\{x : c_i(x) = k\} \cap V_L))}{\frac{L_0}{l}\mu^H(\varphi(V_L))} \\
&= \mu^H(\{c_i = k\}|\varphi(V_L)) \\
&= 2^{-L_i}.
\end{aligned}$$

□

Remark 7 *Using Lemma 4 we obtain*

$$\mu^H(\{c_i = k_0, \ldots, c_{i+m} = k_m\}|l_i, \ldots, l_{i+m}) = \begin{cases} 2^{-\sum_{j=i}^{i+m} l_j} & \text{if } \forall i \leq j \leq i+m,\ c_j < 2^{l_j}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6 *The shift measurable entropy of $h_{\mu_F}(\sigma)$ is positive.*

Proof Let α be the partition of Ω by the coordinate 0 and denote by $\alpha_{-n}^n(x)$ the element of the partition $\alpha \vee \sigma^{-1}\alpha ... \vee \sigma^{-n+1} \vee \sigma^1 ... \vee \sigma^n\alpha$ which contains x. It follows from the definition of μ^F, that $\mu^F(\alpha_{-n}^n(x)) \leq \nu\left(l(y) \in \mathbb{N}^{\mathbb{Z}} : y \in \alpha_{-n}^n(x)\right)$. For all $x \in \Omega$ there exist sequences of positive integers $(-N_i^+)_{i\in\mathbb{N}}$ and $(N_i^-)_{i\in\mathbb{N}}$ such that $x_{-N_i^+} = x_{N_i^-} = E$. Let $K = \frac{\nu^3}{1-\nu} = \sum_{j>2} \nu^j$. If $y \in \Omega \cap \alpha_{-N_i^-}^{N_i^+}(x)$ then

$$\nu\left(l(y)\right) = \nu_*(l_0)\pi_{k=1}^i \left(\nu_*(l_{-k})\nu_*(l_k)\right) = \nu^{N_i^- + N_i^+ + 1}/(K^{2i+1}).$$

Since $l_i \geq 3$ and $\sum_{k=-i}^i l_k = N_i^- + N_i^+$, we obtain

$$\log\left(\nu(l(y))\right) = (N_i^- + N_i^+ + 1)\log\nu - (2i+1)\log K \leq (N_i^- + N_i^+ + 1)\log\frac{1}{1-\nu}.$$

Since $\mu^F(\Omega) = 1$ we get

$$\lim_{i\to\infty} \int_{A^{\mathbb{Z}}} \frac{-\log(\mu^F(\alpha_{-N_i^-}^{N_i^+}(x)))}{N_i^- + N_i^+ + 1} d\mu^F(x) \geq (\log\frac{1}{1-\nu}) > 0.$$

By the probabilistic version of the Shannon-McMillan-Breiman theorem for a $\mathbb{Z}$-action [5], the left side of the previous inequality is equal to $h_\mu^F(\sigma)$, so we can conclude. □

8 Lyapunov exponents

Recall that no information can cross a counter before it reaches the top, that is, before time $\frac{1}{2}(\overline{c}_i - c_i)$ since at each step the counter is incremented by 1 or 2. When the information reaches the next counter, it has to wait more than $\frac{1}{2}(\overline{c}_{i+1} - c_{i+1})$, where this quantity is estimated at the arrival time of the information, and so on.

But each counter is uniformly distributed among its possible values. So expectation of these times is bounded below by $\frac{1}{4}\mathbb{E}[\overline{c}_i] = \mathbb{E}[2^{l_i-2}]$. A good choice of the measure ν can make the expectation of the l_i finite so that we can define the invariant measure μ^F. But $\mathbb{E}[2^{l_i}]$ is infinite so that expectation of time needed to cross a counter is infinite and hence the sum of these times divided by the sum of the length of the binary counters tends to infinity.

Instead of being so specific, we use a rougher argument. Taking ν_* to be geometric, we show that there exists a counter large enough to slow the speed of transmission of information. That is, for given n, there is a counter of length larger than $2\ln n$, information needs a time of order n^δ, with $\delta > 1$ to cross. This is enough to conclude.

Proposition 7 *We have*

$$I^+_{\mu^F} + I^-_{\mu^F} = 0.$$

Proof We just have to show that $I^+_{\mu^F} = 0$ since $I^-_{\mu^F}$ is clearly zero. Recall that $I^+_n(x)$ is the minimal number of coordinates that we have to fix to ensure that for all y such that as soon as $y(-I^+_n(x), \infty) = x(-I^+_n(x), \infty)$, we have $F^k(y)(0, \infty) = F^k(x)(0, \infty)$ for all $0 \le k \le n$. Let

$$t_F(n)(x) = \min\{s \ : \ \exists y, y(-n, \infty) = x(-n, \infty) \text{ and } F^s(x)(0, \infty) \neq F^s(y)(0, \infty)\}$$

be the time needed for a perturbation to cross n coordinates. Note that $t_F(n)$ and I^+_n are related by $t_F(s)(x) \ge n \Leftrightarrow I^+_n(x) \le s$. We now define an analog of $t_F(n)$ for the model. For all $x \in \varphi(\Omega)$, let $M_n(x)$ be the smallest negative integer m such that $\sum_{i=m}^{0} l_i(x) \le n$. Define

$$t_H(n)(x) = \min\left\{ s \ge 0 : \ \exists y \in \Omega, \quad \begin{array}{l} y(M_n(x), \infty) = x(M_n(x), \infty), \\ H^s(x)(0, \infty) \neq H^s(y)(0, \infty) \end{array} \right\}.$$

Define for all large enough positive integer, the subset of Ω

$$U_n = \left\{ x : \quad \begin{array}{l} l_0(x) \le 2\ln(n), \\ \exists i \in \{M_n(x), \dots, -1\}, l_i(x) \ge 2\ln(n), \\ c_i(x) \le 2^{l_i(x)} - 2^{1.5\ln(n)} \end{array} \right\}.$$

We claim that
$$\lim_{n\to\infty} \mu^F(U_n) = 1.$$
Choose and fix an integer n large. Note that if $\forall i \in \{M_n(x),\dots,0\}$, $l_i \leq 2\ln(n)$, we have $|M_n(x)| \geq \frac{n}{2\ln(n)}$. Write $M = -\lfloor \frac{n}{2\ln(n)} \rfloor$ and define
$$V_n = \left\{ L \in \mathbb{N}^{|M|+1} : L_0 \leq 2\ln n \text{ and } \forall i \in \{M,\dots,-1\}, L_i \leq 2\ln(n) \right\}.$$
Note that on $\{l_0 \leq 2\ln n\}$, existence of $i \geq M$ with $l_i \geq 2\ln n$ yields existence of $i \geq M_n(x) \geq M$ with $l_i \geq 2\ln n$ (the same i).

Given a $L = (L_M,\dots,L_0)$ in the complementary of V_n denoted by V_n^c, we denote $V_L^\Omega = \{x \in \Omega : (l_M(x),\dots,l_0(x)) = L\}$ and we denote $i(L)$ the larger index $M < i < 0$ with $L_i > 2\ln n$. Let V_n^{c*} be the subset of V_n^c with $L_0 \leq 2\ln(n)$. The measure of U_n is bounded below by
$$\begin{aligned}
\mu^F(U_n) &= \sum_{L\in\mathbb{N}^M} \mu^F(U_n \cap V_L^\Omega) \\
&\geq \sum_{L\in V_n^{c*}} \mu^F(U_n \cap V_L^\Omega\}) \\
&\geq \sum_{L\in V_n^{c*}} \mu^F(\{c_{i(L)}(x) \leq 2^{l_{i(L)}} - 2^{1.5\ln(n)}\} \cap V_L^\Omega) \\
&= \sum_{L\in V_n^{c*}} \mu^F\left(c_{i(L)}(x) \leq 2^{l_{i(L)}} - 2^{1.5\ln(n)} \,|\, V_L^\Omega\right) \mu^F\left(V_L^\Omega\right).
\end{aligned}$$
According to Lemma 4, given the l_i's, for $M \leq i \leq 0$, the random variable c_i is uniformly distributed. Hence for all $L \in V_n^{c*}$,
$$\mu^F(\{x|c_i(x) \geq 2^{l_i} - 2^{1.5\ln(n)}\}|V_L^\Omega) = 2^{1.5\ln(n)}2^{-L_i(L)} \leq 2^{1.5\ln(n)-2\ln(n)} = n^{-0.5\ln(2)},$$
so that
$$\mu^F(U_n) \geq \sum_{L\in V_n^{c*}} (1-n^{-0.5\ln(2)})\mu^F\left(V_L^\Omega\right) \geq (1-n^{-0.5\ln(2)})\nu(V_n^{c*}).$$
Since $\nu_*(l_i \geq 2\ln(n)) = cst.\sum_{k\geq 2\ln(n)} \nu^k \leq cst.n^{2\ln\nu}$, and the l_i's are independent, it is straightforward to prove the existence of constants c_1, c_2 and c_3 independent on n, such that,
$$\nu(V_n) \leq \left(1 - \sum_{k\geq 2\ln(n)} \nu^k\right)^{|M|} \leq c_1 n^{2\ln\nu} + c_2 \exp\left(-c_3 \frac{n^{1-2\ln(\nu)}}{2\ln(n)}\right),$$
and $\nu(V_n^{c*}) \geq (1-\nu(V_n))\nu_*(l_0 \leq 2\ln(n))$.

We conclude that

$$\mu^F(U_n) \geq \left(1-n^{-0.5\ln(2)}\right)(1-c_4 n^{2\ln\nu})\left(1-c_1 n^{2\ln\nu}+c_2\exp\left(-c_3\frac{n^{1-2\ln(\nu)}}{2\ln(n)}\right)\right).$$

Since $\nu=\frac{2}{3}$, $2\ln(\nu)<1$, this bound converges to 0 and the claim follows.

Now, we claim that there is a constant $c>0$ and a constant $\delta>1$ such that on U_n, we have

$$t_H(n)(x) \geq cn^\delta.$$

Indeed, if $x\in U_n$, and y is such that $y(-n,\infty)=x(-n,\infty)$, then $M(x,n)=M(y,n)=:M$ and $\varphi(y)(-M,\infty)=\varphi(x)(-M,\infty)$. Hence there is an index $0<i\leq M$ with $l_i(x)=l_i(y)=L$ and $c_i(x)=c_i(y)=c$ satisfying

$$L\geq 2\ln(n) \text{ and } c\leq 2^L-2^{1.5\ln(n)}.$$

Notice that $i<0$ because $x\in U_n$. For all $s\leq\frac{1}{2}2^{1.5\ln(n)}$, we have $r_i^s(x)=r_i^s(y)=0$, since $c_i^s(x)<2^{l_i(x)}$. Hence, for all $j>i$ (and in particular for $j=0$), $c_j^s(x)=c_j^s(y)$. For $j=0$, this implies that $t_H(n)(x)\geq\frac{1}{2}n^{1.5\ln 2}$. As $1.5\ln(2)>1$ the claim holds.

There is no y with $y(-n,\infty)=x(-n,\infty)$ and $F^s(x)(0,\infty)\neq F(y)(0,\infty)$ if $s<t_H(n)(x)$ because $y(-n,\infty)=x(-n,\infty)\Rightarrow\varphi(y)(-M(x,n),\infty)=\varphi(x)(-M(x,n),\infty)$. This implies that $t_F(n)(x)\geq t_H(n)(x)$. It follows that $t_F(n)(x)\geq cn^\delta$ on U_n.

Setting $s=\left[n^{\frac{1}{\delta}}\right]$, we see that $t_F(\left[n^{\frac{1}{\delta}}\right])\geq n \Leftrightarrow I_n^+\leq\left[n^{\frac{1}{\delta}}\right]$. We deduce that there is a constant c such that, on U_n,

$$\frac{I_n^+(x)}{n}\leq cn^{\frac{1}{\delta}-1}.$$

Since $\frac{I_n^+(x)}{n}$ is bounded (by $r=2$, radius of the automaton), the conclusion follows from the inequality

$$\int\frac{I_n^+}{n}d\mu^F\leq\int_{U_n}c\,n^{\frac{1}{\delta}-1}d\mu^F+2\mu^F(U_n^c).$$

That is,

$$I_{\mu^F}^+=0.$$

□

Using Proposition 3 we obtain

Corollary 1 *The measurable entropy $h_{\mu^F}(F)$ is equal to zero.*

9 Questions

We end this paper with a few open questions and conjectures.

Conjecture 1 *The measure μ_F is shift-ergodic.*

The measure μ^F is clearly not F-ergodic since the Es do not move. It is still not clear to us whether or not it is possible to construct a sensitive automaton with null Lyapunov exponents for a F-ergodic measure.

Conjecture 2 *A sensitive cellular automaton acting surjectively on an irreducible subshift of finite type has average positive Lyapunov exponents if the invariant measure we consider is the Parry measure on this subshift.*

Conjecture 3 *If a cellular automaton has no equicontinuous points (i.e. it is sensitive), then there exists a point x such that $\liminf \frac{I_n^+(x)}{n} > 0$ or $\liminf \frac{I_n^-(x)}{n} > 0$.*

10 References

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